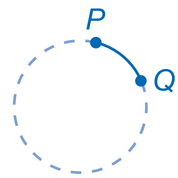
LESSON 25

Circles, Angles, and Arcs

Measuring Arcs and Angles

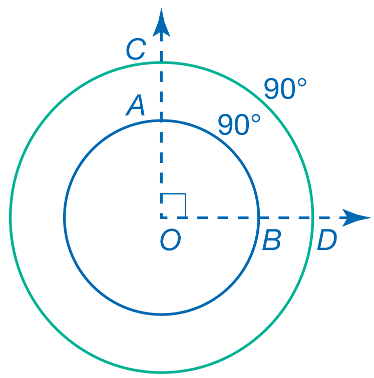
UNDERSTAND An **arc** is an unbroken part of a circle. An arc contains two endpoints and all the points on a circular curve between those points. The name of an arc contains its endpoints covered by an arc-like symbol, such as $\overset{\frown}{PQ}$. Sometimes, another point on the arc is included in the name.



An arc can be measured in two ways: by the length along its curve and by the measure of its **central angle**. A central angle is an angle whose vertex is the center of a circle. The rays of a central angle pass through the circle and cut off an arc called the **intercepted arc**. An intercepted arc and its central angle have the same measure. The measure of an arc is indicated by placing the letter m before the arc name, such as $m\overset{\frown}{PQ}$.

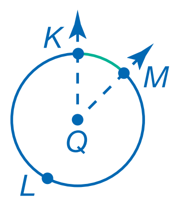
The blue circle and green circle on the right are **concentric circles** because they have the same center. $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ have the same central angle, $\angle COD$, and thus same measure, even though $\overset{\frown}{CD}$ is longer than $\overset{\frown}{AB}$.

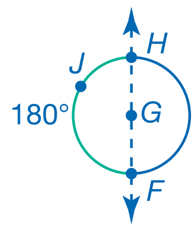
$$m\overset{\frown}{AB} = m\overset{\frown}{CD} = m\angle COD$$



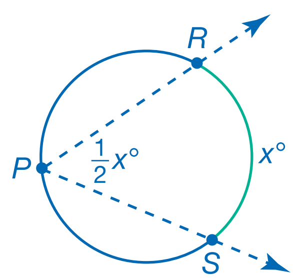
If the central angle increases, the measure of both arcs will likewise increase. If $\overrightarrow{OD}$ is rotated clockwise to lie on top of $\overrightarrow{OC}$, the angle will measure 360°, so one full circle measures 360°.

Circular arcs are classified according to their measure. **Minor arcs** measure less than 180°. They are typically named using only two points. The green minor arc in circle Q on the right is called $\overset{\frown}{KM}$. **Major arcs** measure more than 180°. They are sometimes named using three points. The blue major arc in circle Q is named $\overset{\frown}{KLM}$. An arc measuring exactly 180° may be called a **semicircle**. In circle G on the right, $\overset{\frown}{FJH}$ is a semicircle and $\overline{FH}$ is a diameter.



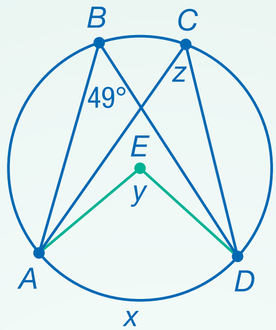


UNDERSTAND An **inscribed angle** has a vertex on the circle and has rays that contain chords of the circle. An inscribed angle will intercept an arc, just as a central angle does. In the circle on the right, inscribed angle RPS intercepts $\overset{\frown}{RS}$. The measure of an inscribed angle is equal to half the measure of its intercepted arc.



Connect

In Circle E, $m\angle ABD = 49°$. Find the measures of $\overset{\frown}{AD}$, $\angle AED$, and $\angle ACD$.



1 Determine the measure of $\overset{\frown}{AD}$.

Let x be the measure of $\overset{\frown}{AD}$.

$x = m\overset{\frown}{AD}$

Angle ABD is an inscribed angle that intercepts $\overset{\frown}{AD}$. So:

$$m\angle ABD = \frac{1}{2} \cdot m\overset{\frown}{AD}$$
$$49° = \frac{1}{2} \cdot x$$
$$98° = x$$

▶ $m\overset{\frown}{AD} = 98°$

2 Find the measure of $\angle AED$.

Let y be the measure of $\angle AED$.

$y = m\angle AED$

Angle AED is a central angle that intercepts $\overset{\frown}{AD}$. So:

$$m\angle AED = m\overset{\frown}{AC}$$
$$y = 98°$$

▶ $m\angle AED = 98°$

3 Find the measure of $\angle ACD$.

Let z be the measure of $\angle ACD$.

$z = m\angle ACD$

Angle ACD is an inscribed angle that intercepts $\overset{\frown}{AD}$. So:

$$m\angle ACD = \frac{1}{2} \cdot m\overset{\frown}{AD}$$
$$z = \frac{1}{2} \cdot 98°$$
$$z = 49°$$

▶ $m\angle ACD = 49°$

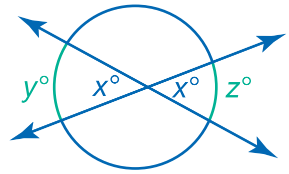
DISCUSS

Based on this example, what can you conclude about two inscribed angles that intercept the same arc? What can you conclude about an inscribed angle and a central angle that intercept the same arc?

Angles Formed by Intersecting Lines

UNDERSTAND When lines and line segments intersect inside a circle, they can form angles that are neither central angles (because their vertexes do not fall on the circle's center) nor inscribed angles (because their vertexes do not fall on one of the circle's points). However, the measures of these angles can be determined if the measures of the arcs they cut off are known.

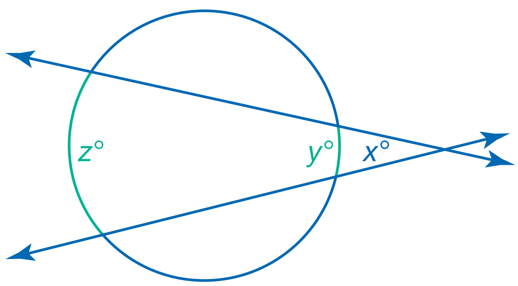
Recall that a pair of non-adjacent angles formed by two intersecting lines are called vertical angles and that vertical angles are always congruent. When two chords or two secant lines intersect inside a circle, the measure of both vertical angles formed is equal to half the sum of the measures of the intercepted arcs.



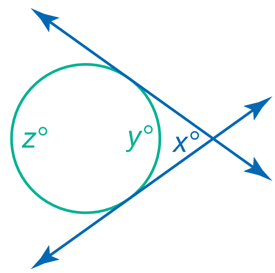
$$x° = \frac{1}{2}(y° + z°)$$

UNDERSTAND Secant lines and tangent lines can intersect outside a given circle. When two such lines intersect outside a circle, the measure of the angle at which they intersect is equal to half the difference of the measures of the intercepted arcs.

Two Secant Lines

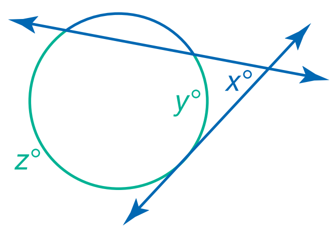


Two Tangent Lines

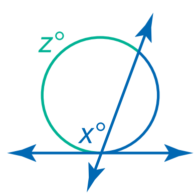


In each of these three diagrams
$x° = \frac{1}{2}(z° - y°)$.

A Secant Line and a Tangent Line



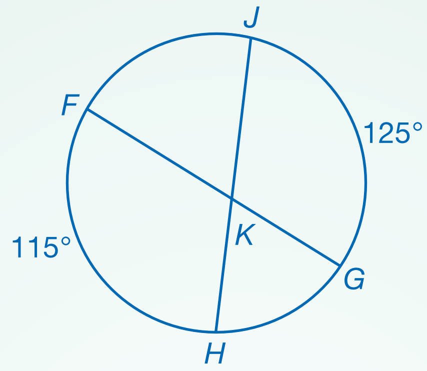
In the case where a tangent line and a secant line intersect, imagine shrinking the smaller arc until $y = 0$. This would produce a special case in which the secant line intersects the tangent line at the point of tangency. Substituting 0 for y in the formula above tells us that the measure of the angle formed is equal to half the measure of its intercepted arc.



$$x° = \frac{1}{2}z°$$

Connect

In the circle, chord $\overline{FG}$ intersects chord $\overline{HJ}$ at point K. What are the measures of the four angles formed by the intersection of the chords?



1 Relate the angles to one another.

Angles *FKH* and *JKG* are vertical angles, so they are congruent.

Angles *FKJ* and *HKG* are also vertical angles, so they are congruent.

Every pair of adjacent angles, such as $\angle FKH$ and $\angle HKG$, form a linear pair, so adjacent angles are supplementary.

2 Find the measure of $\angle FKH$ and $\angle JKG$.

Arc *FH* and arc *GJ* are intercepted by vertical angles *FKH* and *JKG*.

$m\angle FKH = \frac{1}{2}(m\overset{\frown}{FH} + m\overset{\frown}{GJ})$

$m\angle FKH = \frac{1}{2}(115 + 125)$

$m\angle FKH = \frac{1}{2}(240)$

$m\angle FKH = 120°$

Because angles *FKH* and *JKG* are congruent, they have the same measure.

▶ The measures of $\angle FKH$ and $\angle JKG$ are 120°.

3 Find the measure of $\angle FKJ$ and $\angle HKG$.

Because angles *FKH* and *HKG* are supplementary, they sum to 180°.

$m\angle FKH + m\angle HKG = 180°$

$120° + m\angle HKG = 180°$

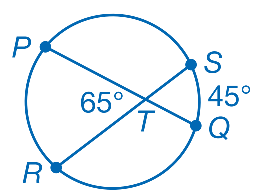
$m\angle HKG = 60°$

Because angles *FKJ* and *HKG* are congruent, they have the same measure.

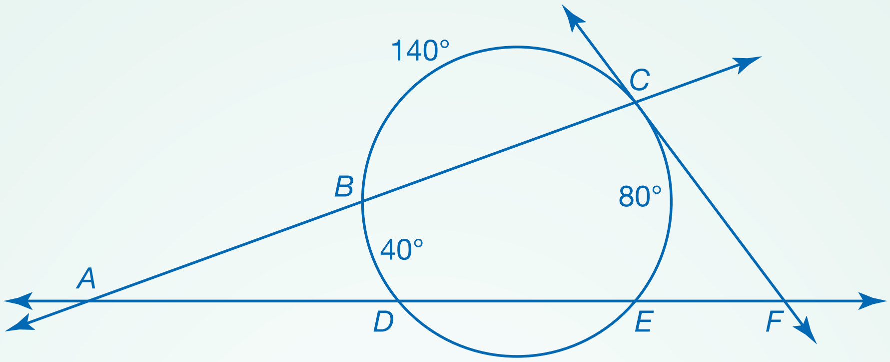
▶ The measures of $\angle FKJ$ and $\angle HKG$ are 60°.

TRY

What is the measure of $\overset{\frown}{PR}$?



EXAMPLE A Secants $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ intersect at point A. $\overleftrightarrow{CF}$ is tangent to the circle at point C and intersects $\overleftrightarrow{DE}$ at point F. What are the measures of $\angle BAD$ and $\angle CFE$?



1 Find the measure of $\angle BAD$.

Angle BAD results from the intersection of two secant lines, so its measure is equal to half the difference of its intercepted arcs.

Its intercepted arcs are $\overset{\frown}{BD}$ and $\overset{\frown}{CE}$.

$$m\angle BAD = \frac{1}{2}(m\overset{\frown}{CE} - m\overset{\frown}{BD})$$

$$m\angle BAD = \frac{1}{2}(80° - 40°)$$

$$m\angle BAD = \frac{1}{2}(40°)$$

▶ $m\angle BAD = 20°$

2 Find the measure of $\angle CFE$.

Angle CFE results from the intersection of a secant line, $\overleftrightarrow{DE}$, and a tangent line, $\overleftrightarrow{CF}$, so its measure is equal to half the difference of its intercepted arcs.

Its intercepted arcs are $\overset{\frown}{CE}$ and $\overset{\frown}{CD}$. $\overset{\frown}{CD}$ is divided into $\overset{\frown}{CB}$ and $\overset{\frown}{BD}$.

$$m\overset{\frown}{CD} = m\overset{\frown}{CB} + m\overset{\frown}{BD}$$

$$m\overset{\frown}{CD} = 140° + 40°$$

$$m\overset{\frown}{CD} = 180°$$

$$m\angle CFE = \frac{1}{2}(m\overset{\frown}{CD} - m\overset{\frown}{CE})$$

$$m\angle CFE = \frac{1}{2}(180° - 80°)$$

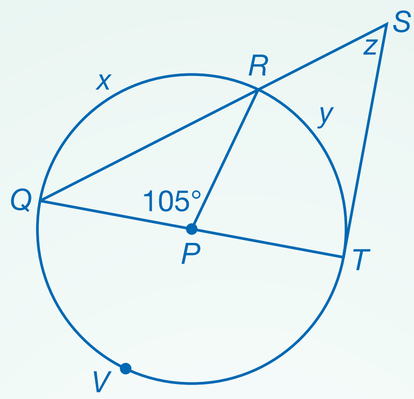
$$m\angle CFE = \frac{1}{2}(100°)$$

▶ $m\angle CFE = 50°$

TRY

What is the measure of $\angle ACF$?

EXAMPLE B In the diagram below, $\overline{ST}$ is tangent to circle P at point T. Find the values of x, y, and z.



1 Find the value of x.

The measure of an arc is equal to the measure of its central angle. The central angle of $\overset{\frown}{QR}$ is $\angle QPR$.

▶ $m\angle QPR = 105°$, so $x = 105°$.

2 Find the value of y.

$\overline{QT}$ is a diameter of circle P, so $\overset{\frown}{QRT}$ is a semicircle that measures 180°.

The measure of $\overset{\frown}{QRT}$ is equal to the sum of the measures of $\overset{\frown}{QR}$ and $\overset{\frown}{RT}$.

$m\overset{\frown}{QR} + m\overset{\frown}{RT} = m\overset{\frown}{QRT}$

$105° + y = 180°$

▶ $y = 75°$

3 Find the value of z.

Secant line SQ and tangent line ST intersect outside the circle at point S. They intercept arcs QVT and RT.

Because $\overline{QT}$ is a diameter of circle P, $\overset{\frown}{QVT}$ is a semicircle that measures 180°.

$m\angle QST = \frac{1}{2}(m\overset{\frown}{QVT} - m\overset{\frown}{RT})$

$z = \frac{1}{2}(180 - 75)$

$z = \frac{1}{2}(105)$

▶ $z = 52.5°$

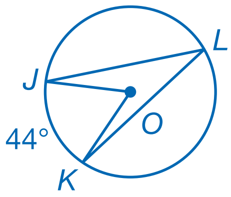
TRY

What is the measure of $\angle RQT$ in the circle?

Practice

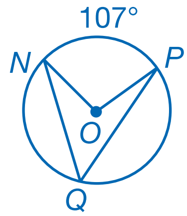
Identify the measure of the angles in each circle *O*.

1.



$m\angle JOK =$ ______ $m\angle JLK =$ ______

2.

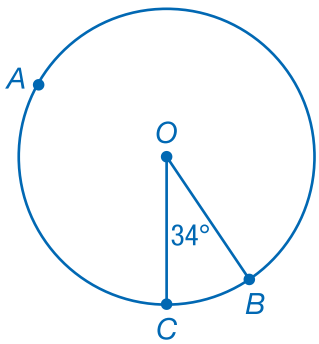


$m\angle NOP =$ ______ $m\angle NQP =$ ______

HINT Use what you know about central angles and inscribed angles.

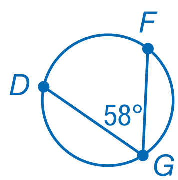
Identify the measure of the arcs.

3.



$m\overset{\frown}{CB} =$ ______ $m\overset{\frown}{CAB} =$ ______

4.



$m\overset{\frown}{DF} =$ ______ $m\overset{\frown}{DGF} =$ ______

REMEMBER A full circle measures 360°.

Write *true* or *false* for each statement. If the statement is false, rewrite it so that it is true.

5. An angle whose vertex is on the circle and whose rays contain radii is called a central angle.

____

6. A semicircle is an arc that measures 180°.

____

7. A minor arc has a measure greater than 180°.

____

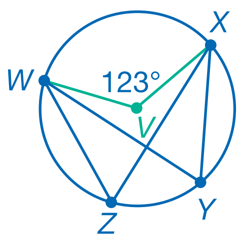
8. The measure of an angle formed by two secant lines that intersect outside a circle is half the sum of the measures of the intercepted arcs.

____

____

Find the measure of each angle or arc in circles *V* and *E*.

9.

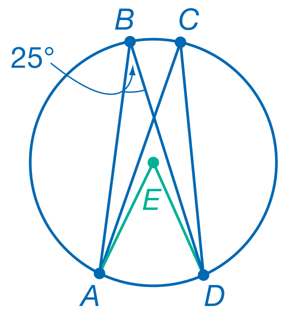


m$\overset{\frown}{WX}$ = ______

m∠*WZX* = ______

m∠*WYX* = ______

10.



m∠*ACD* = ______

m$\overset{\frown}{AD}$ = ______

m∠*AED* = ______

Choose the best answer.

11. In circle *O*, angle *TOV* measures 56°. What is the measure of the arc it intercepts, arc *TV*?

A. 28°

B. 56°

C. 112°

D. 304°

12. Points *P*, *Q*, and *R* are on circle *N*. If ∠*PQR* measures 74°, what is the measure of the arc it intercepts, $\overset{\frown}{PR}$?

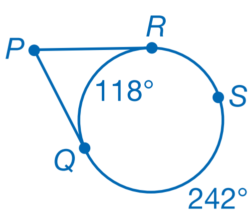
A. 37°

B. 74°

C. 148°

D. 212°

13. Segments $\overline{PR}$ and $\overline{PQ}$ are tangent to the circle below. Which expression is equivalent to the measure of ∠*QPR*?



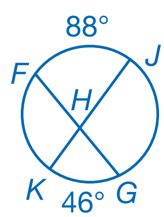
A. $\frac{1}{2}(118°)$

B. $\frac{1}{2}(242°)$

C. $\frac{1}{2}(242° - 118°)$

D. $\frac{1}{2}(242° + 118°)$

14. Chords $\overline{FG}$ and $\overline{JK}$ intersect at point *H*. Which expression is equivalent to the measure of ∠*FHJ*?



A. $\frac{1}{2}(46°)$

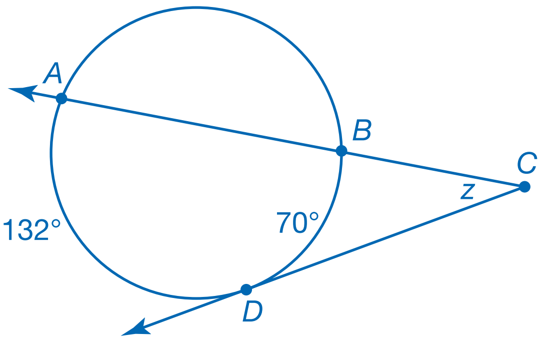
B. $\frac{1}{2}(88°)$

C. $\frac{1}{2}(88° - 46°)$

D. $\frac{1}{2}(88° + 46°)$

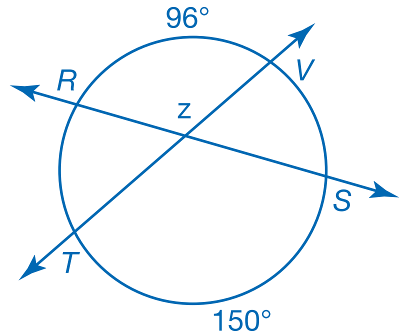
Find the value of *z*.

15.



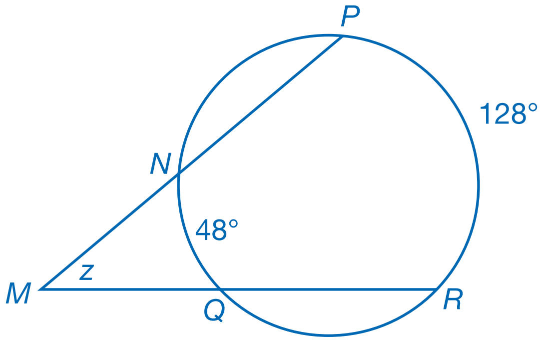
$z =$ ______

16.



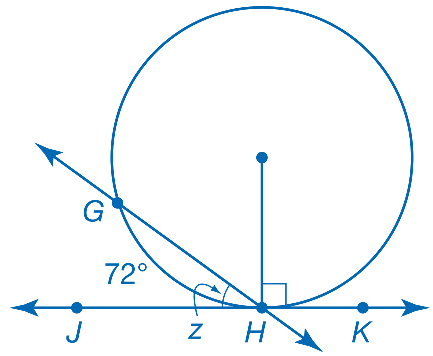
$z =$ ______

17.



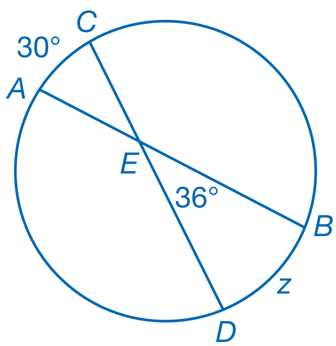
$z =$ ______

18.



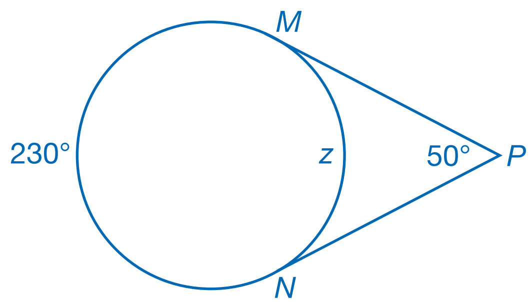
$z =$ ______

19.



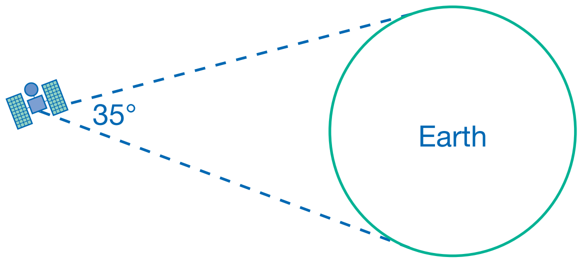
$z =$ ______

20.



$z =$ ______

21. **SHOW** A satellite in orbit above Earth's equator has a camera with a 35° viewing angle of Earth. What is the measure of the arc of the equator that can be viewed from the satellite? Explain how you found your answer.



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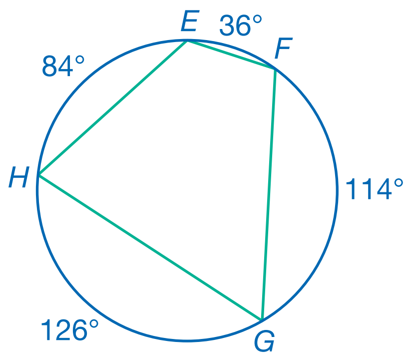
22. **PROVE** Quadrilateral *EFGH* is said to be inscribed in the circle below because all of its vertices lie on the circle. Fill in the blanks based on this figure.



$m\widehat{EF} + m\widehat{FG} + m\widehat{GH} + m\widehat{HE} =$ ______°

$m\angle E = \frac{1}{2}(m\widehat{FG} + m$______$) =$ ______°

$m\angle F = \frac{1}{2}(m\widehat{GH} + m$ ______$) =$ ______°

$m\angle G = \frac{1}{2}(m$______ $+ m$______$) =$ ______°

$m\angle H = \frac{1}{2}(m$______ $+ m$______$) =$ ______°

$m\angle E + m\angle G =$ ______°

$m\angle F + m\angle H =$ ______°

In a quadrilateral inscribed in a circle, opposite angles are ____________________.